Fibonacci vectors.

Problem with a solution proposed by Arkady Alt , San Jose , California, USA

Let $(\bar{\mathbf{a}}_n)_{n\geq 0}$ be sequence of vectors defined recursively $\bar{\mathbf{a}}_0 = \bar{\mathbf{a}}, \bar{\mathbf{a}}_1 = \bar{\mathbf{b}}$ and $\bar{\mathbf{a}}_{n+1} = \bar{\mathbf{a}}_n + \bar{\mathbf{a}}_{n-1}$.

Prove that for any $n \in \mathbb{N}$ the sum of vectors $\bar{\mathbf{a}}_0 + \bar{\mathbf{a}}_1 + \ldots + \bar{\mathbf{a}}_{4n+1}$ is collinear to one of the summand vectors. Find this vector and coefficient of collinearity.

Solution.

First note that $\bar{\mathbf{a}}_n = \bar{\mathbf{c}}_1 F_{n-1} + \bar{\mathbf{c}}_2 F_n$.

We have $\bar{\mathbf{a}}_0 = \bar{\mathbf{c}}_1 F_{-1} + \bar{\mathbf{c}}_2 F_0 \iff \bar{\mathbf{a}} = \bar{\mathbf{c}}_1, \bar{\mathbf{a}}_1 = \bar{\mathbf{c}}_1 F_0 + \bar{\mathbf{c}}_2 F_1 \iff \bar{\mathbf{b}} = \bar{\mathbf{c}}_2$.

Since $\bar{\mathbf{a}} F_{n-1} + \bar{\mathbf{b}} F_n$ coincide with $\bar{\mathbf{a}}_n$ if $n = 0, 1$ and supposition $\bar{\mathbf{a}}_n = \bar{\mathbf{a}} F_{n-1} + \bar{\mathbf{b}} F_n, \bar{\mathbf{a}}_{n-1} = \bar{\mathbf{a}} F_{n-2} + \bar{\mathbf{b}} F_{n-1}, n \geq 1$ yields $\bar{\mathbf{a}}_{n+1} = \bar{\mathbf{a}} F_{n-1} + \bar{\mathbf{b}} F_n + \bar{\mathbf{a}} F_{n-2} + \bar{\mathbf{b}} F_{n-1} = \bar{\mathbf{a}} F_n + \bar{\mathbf{b}} F_{n+1}$ then by Math. Induction $\bar{\mathbf{a}}_n = \bar{\mathbf{a}} F_{n-1} + \bar{\mathbf{b}} F_n$ for all $n \in \mathbb{N} \cup \{0\}$.

Using such representation of $\bar{\mathbf{a}}_n$ we obtain $\sum_{i=0}^{4n+1} \bar{\mathbf{a}}_i = \bar{\mathbf{a}} \sum_{i=0}^{4n+1} F_{i-1} + \bar{\mathbf{b}} \sum_{i=0}^{4n+1} F_i =$
$\bar{\mathbf{a}} \sum_{i=0}^{4n+1} (F_{i+1} - F_i) + \bar{\mathbf{b}} \sum_{i=0}^{4n+1} (F_{i+2} - F_{i+1}) = \bar{\mathbf{a}} F_{4n+2} + \bar{\mathbf{b}} (F_{4n+3} - 1) =$
$\dfrac{F_{4n+2}}{F_{2n+1}} (\bar{\mathbf{a}} F_{2n+1} + \bar{\mathbf{b}} F_{2n+2}) = \dfrac{F_{4n+2}}{F_{2n+1}} \bar{\mathbf{a}}_{2n+2}$.

Remark.

If we consider sum of any $4n+2$ consecutive terms of this sequence $\bar{\mathbf{a}}_m + \bar{\mathbf{a}}_{m+1} + \ldots + \bar{\mathbf{a}}_{m+4n+1}$ then $\bar{\mathbf{a}}_m + \bar{\mathbf{a}}_{m+1} + \ldots + \bar{\mathbf{a}}_{m+4n+1} = \dfrac{F_{4n+2}}{F_{2n+1}} \bar{\mathbf{a}}_{m+2n+2}$ because $\bar{\mathbf{a}}_{m+n} = \bar{\mathbf{a}}_m F_{n-1} + \bar{\mathbf{a}}_{m+1} F_n$.